NUMERICAL METHODS

Code: UM304

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UNIT I

SOLUTION OF EQUATIONS

Algebraic equation

- Definition:

An algebraic equation is an equation constructed using the operations of $+, -, \times, \div$, and possibly root taking (radicals). Rational functions and polynomials are examples of algebraic functions.

Transcendental Equations

- Transcendental equations in comparison are not algebraic.
- That is, they contain non-algebraic functions and possibly their inverses functions.
- Equations which contain either trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions are examples of non-algebraic functions which are called transcendental functions.

Fundamental Theorem

- If f(x) is real and continuous in the interval from x = a to x = b and f(a) and f(b) have opposite signs, i.e., f(a)f(b)<0, then there is atleast one real root between a and b.
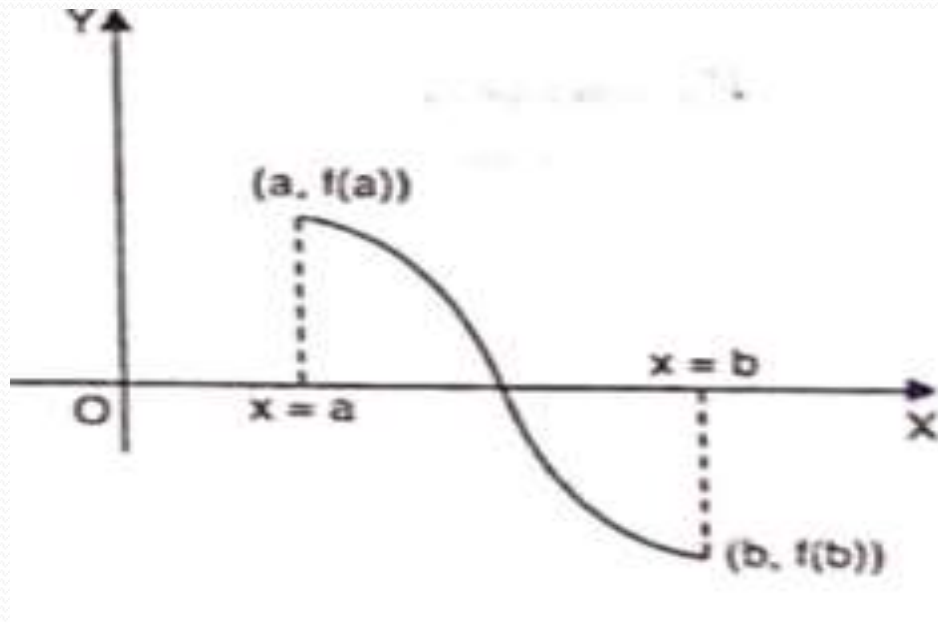
Bisection Method (Bolzano's method)

➢ Interval having method

➢ Suppose we have an equation $f(x) = 0$. Let $f(x)$ be continuous and it can be algebraic or transcendental.

➢ Let the function $f(x)$ changes sign over an interval $x = a$ to $x = b$ (i.e. $f(a)$ and $f(b)$ are of opposite signs or $f(a)f(b) < 0$).

➢ Then there is a root $f(x) = 0$ lying between a and b.

Problems

Find the positive root of $f(x) = x - \cos x$ by bisection method correct to three decimal places.

Solution:

Let $f(x) = x - \cos x$

$$f(0) = 0 - \cos 0 = -1 = \text{-ve}$$

$$f(1) = 1 - \cos 1 = 0.45970 = \text{+ve}$$

Hence the root lies between 0 and 1.

$$x_0 = \frac{0+1}{2} = 0.5$$

$f(0.5) = 0.5 - \cos 0.5 = -0.37758 = \text{-ve}$

So the root lies between 0.5 and 1

$$x_1 = \frac{0.5 + 1}{2} = 0.75$$

f(0.75) = 0.75 – cos(0.75) = 0.018311 = +ve

So the root lies between 0.5 and 0.75

$$x_2 = \frac{0.5 + 0.75}{2} = 0.625$$

f(0.625) = 0.625- cos(0.625) = - 0.18596 = - ve

So the root lies between 0.75 and 0.625

$$x_3 = \frac{0.75 + 0.625}{2} = 0.6875$$

f(0.6875) = 0.6875 – cos(0.6875) = -0.085335 = -ve

So the root lies between 0.6875 and 0.75

$$x_4 = \frac{0.6875 + 0.75}{2} = 0.71875$$

f(0.71875) = 0.71875 – cos(0.71875) = -0.033789 = -ve

So the root lies between 0.71875 and 0.75

$$x_5 = \frac{0.71875 + 0.75}{2} = 0.73438$$

f(0.73438) = 0.73478 – cos(0.73438) = -0.00786 = -ve

So the root lies between 0.73438 and 0.75

$$x_6 = \frac{0.73438 + 0.75}{2} = 0.74219$$

f(0.74219) = 0.74219- cos(0.74219) = 0.00512 = +ve

So the root lies between 0.73438 and 0.74219

$$x_7 = \frac{0.73438 + 0.74219}{2} = 0.73829$$

f(0.73829) = 0.73829 – cos(0.73829) = -0.00133 = -ve
So the root lies between 0.73829 and 0.74219

$$x_8 = \frac{0.73829 + 0.74219}{2} = 0.7402$$

f(0.7402) = 0.7402 – cos(0.7402) = 0.00186 = +ve
So the root lies between 0.7402 and 0.73829

$$x_9 = \frac{0.7402 + 0.73829}{2} = 0.73925$$

f(0.73925) = 0.73925 – cos(0.73925) = 0.000275 = +ve

So the root lies between 0.73829 and 0.73925

$$x_{10} = \frac{0.73829 + 0.73925}{2} = 0.7388$$

$f(x_0)$ approaches the value zero

$$x = 0.738$$

Newton – Raphson method

- Newton's method or method of tangents

- This method is also based on a linear approximation of the function but using a tangent to the curve.

- This method may be developed from the taylor's series expansion.

Newton – Raphson Method (or) Newton's Method (or) Method of Tangents

This method starts with an initial approximation to the root of the equation. A better and closer approximation to the root can be found by using an iterative process.

Newton – Raphson formula:

$$x_{i+1} = x_i - \left[\frac{f(x_i)}{f'(x_i)} \right]; \; i = 0,1,2,\cdots$$

Using Newton – Raphson method, find the root of $x^3 - 6x + 4 = 0$ and correct its 4 decimal points.

Solution

Newton – Raphson formula is

$$x_{i+1} = x_i - \left[\frac{f(x_i)}{f'(x_i)}\right]; \ i = 0,1,2,...$$

Let $f(x) = x^3 - 6x + 4; \ f'(x) = 3x^2 - 6$

$\therefore f(0) = 4$ (positive)

$f(1) = -1$ (negative)

The root lies between 0 and 1.

Let the initial approximation be $x_0 = 0.5$.

Iteration: 1, i = 0

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 0.5 - \left[\frac{(0.5)^3 - 6(0.5) + 4}{3(0.5)^2 - 6}\right] = 0.7143 \ .$$

Iteration: 2, i = 1

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.7143 - \left[\frac{(0.7143)^3 - 6(0.7143) + 4}{3(0.7143)^2 - 6}\right] = 0.7319 \ .$$

Iteration: 3, i = 2

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 0.7319 - \left[\frac{(0.7319)^3 - 6(0.7319) + 4}{3(0.7319)^2 - 6}\right] = 0.7320 \ .$$

Iteration: 4, i = 3

$$x_4 = x_3 - \left[\frac{f(x_3)}{f'(x_3)}\right] = 0.7320 - \left[\frac{(0.7320)^3 - 6(0.7320) + 4}{3(0.7320)^2 - 6}\right] = 0.7320 \ .$$

The value of x is 0.7320.

Evaluate $\sqrt{12}$ applying Newton formula.

Solution: Let $x = \sqrt{12}$

$$\Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$$

Let $f(x) = x^2 - 12 \Rightarrow f'(x) = 2x$

$$f(3) = 9 - 12 = -3\,(-ve)$$

$$f(4) = 16 - 12 = 4\,(+ve)$$

$\therefore$ Root lies between 3 and 4.

and root is closer to 3 since $|f(3)| < |f(4)|$

Take $x_0 = 3$

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

when $n = 0, x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{(3)^2 - 12}{2(3)} = 3.5$

when $n = 1, x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3.5 - \dfrac{f(3.5)}{f'(3.5)} = 3.5 - \dfrac{(3.5)^2 - 12}{2(3.5)} = 3.46429$

when $n = 2, x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 3.46429 - \dfrac{f(3.46429)}{f'(3.46429)} = 3.46429 - \dfrac{(3.46429)^2 - 12}{2(3.46429)} = 3.46410$

when $n = 3, x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 3.46410 - \dfrac{f(3.46410)}{f'(3.46410)} = 3.46410 - \dfrac{(3.46410)^2 - 12}{2(3.46410)} = 3.46410$

Here $x_3 = x_4 = 3.46410$

$\therefore$ The root is 3.46410

REGULA – FALSI MEHOD

- Method of False Position

- Suppose we have an equation f(x) = 0. Let f(a) and f(b) are of opposite signs and let a < b.

- Then the curve y = f(x) meets the x – axis at some point between (a, f(a)) and (b, f(b)).

- The equation of the straight line joining these points which intersects the x – axis is

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}$$

which can be solved for x
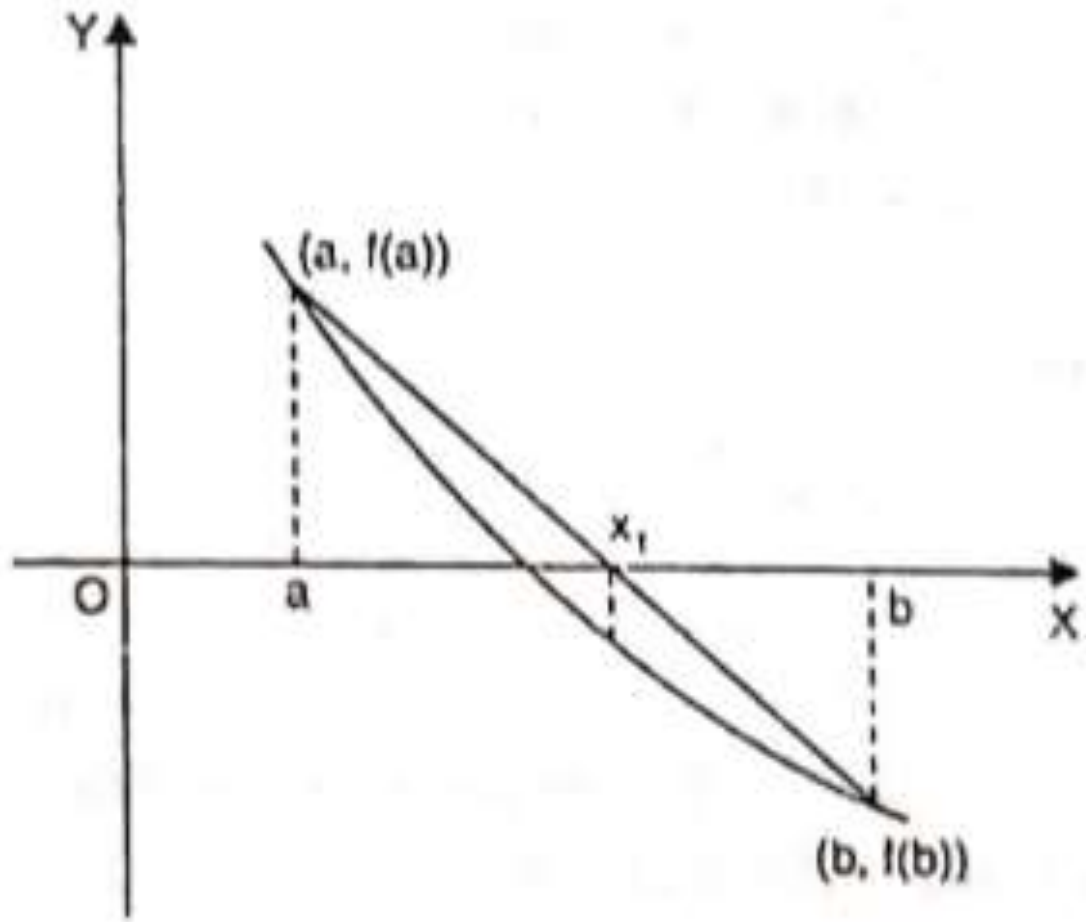
Find the root of $xe^x = 3$ by Regula falsi method correct to 3 decimal places

Solution: Let $f(x) = xe^x - 3$

$$f(1) = -0.2817 \quad < 0$$
and $\quad f(1.1) = 0.3046 \quad > 0$

The root lies between 1 and 1.1.

The approximations are given by

$$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

First approximation:

Taking $a = 1$ and $b = 1.1$

$$x_1 = \frac{(1)(0.3046) + (1.1)(0.2817)}{0.3046 + 0.2817} = 1.048$$

$f(1.048) = -0.011165 < 0$

Second approximation:

Taking a = 1.048 and b = 1.1

$$x_2 = \frac{(1.048)(0.3046) + (1.1)(0.01165)}{0.3046 + 0.011165} = 1.0498$$

f (1.0498) = - 0.000638 < 0

Third approximation:

Taking a = 1.0498 and b = 1.1

$$x_3 = \frac{(1.0498)(0.3046) + (1.1)(0.000638)}{0.305238} = 1.0499$$

f (1-0499) = - 0.000052 , which is very near zero.

∴ The required root is 1.0499

Find a real root of $x^3 - 9x + 1 = 0$ that lies between 2 and 3 by the method of false position, correct to 3 decimal places

Solution: Let $f(x) = x^3 - 9x + 1$

$f(2) = 8 - 18 + 1 = -9 < 0$

$f(2.5) = -5.875 < 0$

and $f(3) = 27 - 27 + 1 > 0$

Hence the root lies between $a = 2.5$ and $b = 3$.

The approximations are given by

$$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

First approximation:
Taking $a = 2.5$ and $b = 3$

$$x_1 = \frac{2.5(1) - 3(-5.875)}{1 + 5.875} = 2.9273$$

and f(2.9273) = - 0.2664<0

Second approximation:
Taking a = 2.9273 and b = 3

$$x_2 = \frac{2.9273\,(1) - 3(-0.2664\ )}{1 + 0.2664} = 2.9423$$

$$f(2.9423) = -\ 0.0088 < 0$$

Third approximation:
Taking a = 2.9423 and b = 3

$$x_3 = \frac{2.9423\,(1) - 3(-0.2664\ )}{1 + 0.0088} = 2.9425$$

$$f(2.9425) = -0.0054 < 0$$

Fourth approximation:
Taking a = 2.9425 and b = 3

$$x_4 = \frac{2.9425\,(1) - 3(-0.0054\ )}{1 + 0.0054} = 2.943$$

$$f(2.943) = 0.003 > 022$$

Since f(2.943) = 0.003 to nearly zero, the required root is 2.94322

System of Linear Equations

➤ **Direct Methods**

    (i). Gauss Elimination Method

    (ii). Gauss Jordan Method

➤ **Iterative Methods**

    (i). Gauss Jacobi Method

    (ii). Gauss Seidel Method

Direct Methods

- The computation can be completed in a finite number of steps resulting in the exact solution.

- The amount of computation involved can be specified in advance.

- The method is independent of the accuracy desired

Iterative Methods

- Begin with an approximate solution and
- Obtain an improved solution with each step of iteration
- But would require an infinite number of steps to obtain an exact solution without round-off errors
- The accuracy of the solution depends on the number of iterations performed.

Gauss Elimination Method

This method is the most effective direct solution technique. In this method, consider the given system of equations to be $AX = B$.

In Gauss elimination method, we start with the augmented matrix $A|B$ (A with B) of the given system and transform it to $U|K$ (upper triangular matrix with k – rows) i.e a matrix in which all elements below the leading diagonal elements are zero by eliminating row operations. Finally, the solution is obtained by back substitution process.

Principles of Gauss Elimination method

$[A, B] \rightarrow$ upper triangular matrix $(U|K)$, then find x, y, z by back substitution process.

Solve the system of equations by Gauss elimination method.
$$10x - 2y + 3z = 23; 2x + 10y - 5z = -33; 3x - 4y + 10z = 41.$$

Solution: Write the given system of equations in augmented matrix form

$$\begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\dfrac{1}{5} & \dfrac{3}{10} & \dfrac{23}{10} \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix} \quad R_1 \rightarrow R_1 \div 10$$

$$\begin{pmatrix} 1 & -\dfrac{1}{5} & \dfrac{3}{10} & \dfrac{23}{10} \\ 0 & \dfrac{52}{5} & -\dfrac{28}{5} & -\dfrac{188}{5} \\ 0 & -\dfrac{17}{5} & \dfrac{91}{10} & \dfrac{341}{10} \end{pmatrix} \quad R_2 \rightarrow R_2 - 2R_1; R_3 \rightarrow R_3 - 3R_1;$$

$$\begin{pmatrix} 1 & -\dfrac{1}{5} & \dfrac{3}{10} & \bigg| & \dfrac{23}{10} \\ 0 & 1 & -\dfrac{7}{13} & \bigg| & -\dfrac{47}{13} \\ 0 & -\dfrac{17}{5} & \dfrac{91}{10} & \bigg| & \dfrac{341}{10} \end{pmatrix} \quad R_2 \to R_2 \div \dfrac{52}{5}$$

$$\begin{pmatrix} 1 & -\dfrac{1}{5} & \dfrac{3}{10} & \bigg| & \dfrac{23}{10} \\ 0 & 1 & -\dfrac{7}{13} & \bigg| & -\dfrac{47}{13} \\ 0 & 0 & \dfrac{189}{26} & \bigg| & \dfrac{567}{26} \end{pmatrix} \quad R_3 \to R_3 + \dfrac{17}{5} R_2$$

$$\begin{pmatrix} 1 & -\dfrac{1}{5} & \dfrac{3}{10} & \bigg| & \dfrac{23}{10} \\ 0 & 1 & -\dfrac{7}{13} & \bigg| & -\dfrac{47}{13} \\ 0 & 0 & 1 & \bigg| & 3 \end{pmatrix} \quad R_3 \to R_3 \div \dfrac{189}{26}$$

Use back substitution to find the solution to the system.

Here $\quad z = 3$

$substitute\ z = 3\ in\ y - \dfrac{7}{13}z = \dfrac{-47}{13}$

$we\ get\ y = -2$

substitute $y = -2\ and\ z = 3\ in$

$x - \dfrac{1}{5}y + \dfrac{3}{10}z = \dfrac{23}{10}$ we get $x = 1$

Solve the system of equations

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

using Gauss – elimination method.

Solution

The given system of equation is equivalent to $AX = B$ where

$$A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad and \quad B = \begin{pmatrix} 32 \\ 24 \\ 35 \end{pmatrix}$$

$$\therefore (A \mid B) = \begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 28 & 4 & -1 & 32 \\ 2 & 17 & 4 & 35 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 0 & -80 & -281 & -640 \\ 0 & 11 & -16 & -13 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 28R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 0 & -80 & -281 & -640 \\ 0 & 0 & -54.64 & -101 \end{pmatrix} R_3 \to R_3 + \frac{11}{80} R_2$$

$$= (U|K).$$

using back substation method,

$-54.64z = -101$

$z = 1.85$

Also,

$-80y - 281z = -640$

$y = 1.5$

Also,

$x + 3y + 10z = 24$

$x = 1.$

$\therefore x = 1; y = 1.5$ and $z = 1.85.$

Gauss Jordan Method

This method is a modification of Gauss – elimination method. Here, we consider the given system of equations to be AX = B. In Gauss – Jordan method, we start with the augmented matrix (A|B) of the given system of equations and transform it to diagonal matrix of unit matrix by elementary row operations. Finally, the solution is obtained directly without back substitution process.

i.e. $(A\,|\,B) \xrightarrow{\;GJM\;} (D\,|\,K)$ or $(I\,|\,K)$.

Solve the following equations by the Gauss – Jordan method.

$x + 2y + z = 8$; $2x + 3y + 4z = 20$; $4x + y + 2z = 12$.

Solution

The given equations is equivalent to $AX = B$ where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 1 & 2 \end{pmatrix}; \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 \\ 20 \\ 12 \end{pmatrix}$$

$$\therefore (A \mid B) = \begin{pmatrix} 1 & 2 & 1 & 8 \\ 2 & 3 & 4 & 20 \\ 4 & 1 & 2 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 & 8 \\ 0 & -1 & 2 & 4 \\ 0 & -7 & -2 & -20 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 & 16 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -16 & -48 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 7R_2 \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 & 16 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} R_3 \to \dfrac{R_3}{16}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix}$$

$$= (D|K).$$

$$\therefore x = 1; \, y = 2 \text{ and } z = 3.$$

Diagonal system

In the system of linear equations in n unknowns $AX = B$. If the coefficient matrix A is diagonally dominant then the system is said to be a diagonal system.

Thus the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is a diagonal system if

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3|$$

Note : For the Gauss Seidal method to converge quickly, the coefficient matrix A must be diagonally dominant.

If the coefficient matrix A is not diagonally dominant we must rearrange the equations in such a way that the resulting matrix becomes dominant, and then only we can apply Gauss Seidel method.

Iterative Methods of Gauss Jacobi

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \tag{1}$$

$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \tag{2}$$

$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \tag{3}$$

Step 3: Start with the initial values $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ for x, y, z and get $x^{(1)}, y^{(1)}, z^{(1)}$

∴ (1),(2),(3) become

$$x^{(1)} = \frac{1}{a_{11}} \left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)} \right)$$

$$y^{(1)} = \frac{1}{a_{22}} \left(b_2 - a_{21}x^{(0)} - a_{23}z^{(0)} \right)$$

$$z^{(1)} = \frac{1}{a_{33}} \left(b_3 - a_{31}x^{(0)} - a_{32}y^{(0)} \right)$$

Step 4: Using this $x^{(1)}$ for x, $y^{(1)}$ for y, $z^{(1)}$ for z in (1),(2),(3) respectively, we get

$$x^{(2)} = \frac{1}{a_{11}} \left(b_1 - a_{12}y^{(1)} - a_{13}z^{(1)} \right)$$

$$y^{(2)} = \frac{1}{a_{22}} \left(b_2 - a_{21}x^{(1)} - a_{23}z^{(1)} \right)$$

$$z^{(2)} = \frac{1}{a_{33}} \left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)} \right)$$

Continuing in the same procedure until the convergence is confirmed.

The general iterative formula of Gauss-Jacobi is

$$x_1^{(j+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_2^{(j+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_3^{(j+1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j)} + a_{32}x_2^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$\vdots$$

$$x_n^{(j+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j)} + a_{n2}x_2^{(j)} + \cdots + a_{n,n-1}x_{n-1}^{(j)}\right)\right]$$

Solve the given system by Gauss –Jacobi method.
$$8x - 3y + 2z = 20; 4x + 11y - z = 33; 6x + 3y + 12z = 35.$$

Solution:

Iteration	$x = \dfrac{1}{8}(20 + 3y - 2z)$	$y = \dfrac{1}{11}(33 - 4x + z)$	$z = \dfrac{1}{12}(35 - 6x - 3y)$
Initial	$y = 0\ \&\ z = 0$	$x = 0\ \&\ z = 0$	$x = 0\ \&\ y = 0$
I	$x = \dfrac{1}{8}(20 + 3(0) - 2(0))$ $x = 2.5$	$y = \dfrac{1}{11}(33 - 4(0) + (0))$ $y = 3$	$z = \dfrac{1}{12}(35 - 6(0) - 3(0))$ $z = 2.917$

II	$y = 3$ & $z = 2.917$ $$x = \frac{1}{8}(20 + 3(3) - 2(2.917))$$ $x = 2.896$	$x = 2.5$ & $z = 2.917$ $$y = \frac{1}{11}(33 - 4(2.5) + 2.917)$$ $y = 2.356$	$x = 2.5$ & $y = 3$ $$z = \frac{1}{12}(35 - 6(2.5) - 3(3))$$ $z = 0.917$
III	$y = 2.356$ & $z = z = 0.917$ $$x = \frac{1}{8}(20 + 3(2.356) - 2(0.917))$$ $x = 3.154$	$x = 2.896$ & $z = 0.917$ $$y = \frac{1}{11}(33 - 4(2.896) + 0.917)$$ $y = 2.030$	$x = 2.896$ & $y = 2.356$ $$z = \frac{1}{12}(35 - 6(2.896) - 3(2.356)$$ $z = 0.880$
IV	$y = 2.030$ & $z = 0.880$ $$x = \frac{1}{8}(20 + 3(2.030) - 2(0.880))$$ $x = 3.041$	$x = 3.154$ & $z = 0.880$ $$y = \frac{1}{11}(33 - 4(3.154) + 0.880)$$ $y = 1.933$	$x = 3.154$ & $y = 2.030$ $$z = \frac{1}{12}(35 - 6(3.154) - 3(2.030)$$ $z = 0.832$
V	$y = 1.933$ & $z = 0.832$ $$x = \frac{1}{8}(20 + 3(1.933) - 2(0.832))$$ $x = 3.017$	$x = 3.041$ & $z = 0.832$ $$y = \frac{1}{11}(33 - 4(3.041) + 0.832)$$ $y = 1.970$	$x = 3.041$ & $y = 1.933$ $$z = \frac{1}{12}(35 - 6(3.041) - 3(1.933)$$ $z = 0.913$

VI	$y = 1.970 \,\&\, z = 0.913$ $x = \dfrac{1}{8}(20 + 3(1.970) - 2(0.913))$ $x = 3.010$	$x = 3.017 \,\&\, z = 0.913$ $y = \dfrac{1}{11}(33 - 4(3.017) + 0.913)$ $y = 1.986$	$x = 3.017 \,\&\, y = 0.913$ $z = \dfrac{1}{12}(35 - 6(3.017) - 3(0.913)$ $z = 0.916$
VII	$y = 1.986 \,\&\, z = 0.916$ $x = \dfrac{1}{8}(20 + 3(1.986) - 2(0.916))$ $x = 3.016$	$x = 3.010 \,\&\, z = 0.916$ $y = \dfrac{1}{11}(33 - 4(3.010) + 0.916)$ $y = 1.989$	$x = 3.010 \,\&\, y = 1.986$ $z = \dfrac{1}{12}(35 - 6(3.010) - 3(1.986)$ $z = 0.915$
VIII	$y = 1.989 \,\&\, z = 0.915$ $x = \dfrac{1}{8}(20 + 3(1.989) - 2(0.915))$ $x = 3.017$	$x = 3.016 \,\&\, z = 0.915$ $y = \dfrac{1}{11}(33 - 4(3.016) + 0.915)$ $y = 1.987$	$x = 3.016 \,\&\, y = 1.986$ $z = \dfrac{1}{12}(35 - 6(3.016) - 3(1.989)$ $z = 0.912$
IX	$y = 1.987 \,\&\, z = 0.912$ $x = \dfrac{1}{8}(20 + 3(1.987) - 2(0.912))$ $x = 3.017$	$x = 3.017 \,\&\, z = 0.912$ $y = \dfrac{1}{11}(33 - 4(3.017) + 0.912)$ $y = 1.987$	$x = 3.017 \,\&\, y = 1.986$ $z = \dfrac{1}{12}(35 - 6(3.017) - 3(1.987)$ $z = 0.912$

Iterative Methods of Gauss Seidel

Step 1:   Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2:   The above system should write into the form

$$x = \frac{1}{a_{11}} (b_1 - a_{12}y - a_{13}z) \tag{1}$$

$$y = \frac{1}{a_{22}} (b_2 - a_{21}x - a_{23}z) \tag{2}$$

$$z = \frac{1}{a_{33}} (b_3 - a_{31}x - a_{32}y) \tag{3}$$

Step 3: Start with the initial values $y^{(0)} = 0, z^{(0)} = 0$ for y, z and get $x^{(1)}$ from the first equation.

$\therefore$ (1) becomes $x^{(1)} = \dfrac{1}{a_{11}} \left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)} \right)$

Step 4: Using this $x^{(1)}$ in (2), we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$, we get

$\therefore$ (2) becomes $y^{(1)} = \dfrac{1}{a_{22}} \left(b_2 - a_{21}x^{(1)} - a_{23}z^{(0)} \right)$

Step 5: Substitute $x^{(1)}, y^{(1)}$ for x, y in the third equation.

$\therefore$ (3) becomes $z^{(1)} = \dfrac{1}{a_{33}} \left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)} \right)$

Step 6: To find the values of unknowns, use the latest available values on the right side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterate values, then the next iteration will be

$$x^{(r+1)} = \frac{1}{a_{11}} \left(b_1 - a_{12}y^{(r)} - a_{13}z^{(r)} \right)$$

$$y^{(r+1)} = \frac{1}{a_{22}} \left(b_2 - a_{21}x^{(r+1)} - a_{23}z^{(r)} \right)$$

$$z^{(r+1)} = \frac{1}{a_{33}} \left(b_3 - a_{31}x^{(r+1)} - a_{32}y^{(r+1)} \right)$$

Step 7: This process of continued until the convergence is confirmed.

The general iterative formula of Gauss-Seidel is

$$x_1^{(j+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_2^{(j+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j+1)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_3^{(j+1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j+1)} + a_{32}x_2^{(j+1)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$\vdots$$

$$x_n^{(j+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j+1)} + a_{n2}x_2^{(j+1)} + \cdots + a_{n,n-1}x_{n-1}^{(j+1)}\right)\right]$$

Note : If either method converges, Gauss-Seidel converges faster than Jacobi.

Solve the system of equations by Gauss-Seidel method
$$27x + 6y - z = 85; 6x + 15y + 2z = 72; x + y + 54z = 110.$$

Solution:

Iteration	$x = \dfrac{1}{27}(85 - 6y + z)$	$y = \dfrac{1}{15}(72 - 6x - 2z)$	$z = \dfrac{1}{54}(110 - x - y)$
Initial	$y = 0 \,\&\, z = 0$	$x = 3.148 \,\&\, z = 0$	$x = 3.148 \,\&\, y = 3.541$
I	$x = \dfrac{1}{27}(85 - 6(0) + (0))$ $x = 3.148$	$y = \dfrac{1}{15}(72 - 6(3.148) - 2(0))$ $y = 3.541$	$z = \dfrac{1}{54}(110 - 3.148 - 3.541)$ $z = 1.913$
II	$y = 3.541 \,\&\, z = 1.913$ $x = \dfrac{1}{27}(85 - 6(3.541) + (1.913))$ $x = 2.432$	$x = 2.432 \,\&\, z = 1.913$ $y = \dfrac{1}{15}(72 - 6(2.432) - 2(1.913))$ $y = 3.572$	$x = 2.432 \,\&\, y = 3.572$ $z = \dfrac{1}{54}(110 - 2.432 - 3.572)$ $z = 1.926$

III	$y = 3.572 \& z = 1.926$	$x = 2.426 \& z = 1.926$	$x = 2.426 \& y = 3.573$
	$x = \frac{1}{27}\big(85 - 6(3.572) + (1.926)\big)$	$y = \frac{1}{15}\big(72 - 6(2.426) - 2(1.926)\big)$	$z = \frac{1}{54}(110 - 2.426 - 3.573)$
	$x = 2.426$	$y = 3.573$	$z = 1.926$
IV	$y = 3.572 \& z = 1.926$	$x = 2.426 \& z = 1.926$	$x = 2.426 \& y = 3.573$
	$x = \frac{1}{27}\big(85 - 6(3.573) + (1.926)\big)$	$y = \frac{1}{15}\big(72 - 6(2.426) - 2(1.926)\big)$	$z = \frac{1}{54}(110 - 2.426 - 3.573)$
	$x = 2.426$	$y = 3.573$	$z = 1.926$

$\therefore x = 2.426; y = 3.573; z = 1.926$

Unit II
FINITE DIFFERENCES

Finite Differences Operators

For a function y=f(x), it is given that $y_0, y_1, ..., y_n$ are the values of the variable y corresponding to the equidistant arguments, $x_0, x_1, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, ..., x_n = x_0 + nh$. In this case, even though Lagrange and divided difference interpolation polynomials can be used for interpolation, some simpler interpolation formulas can be derived. For this, we have to be familiar with some finite difference operators and finite differences, which were introduced by Sir Isaac Newton. Finite differences deal with the changes that take place in the value of a function f(x) due to finite changes in x. Finite difference operators include, forward difference operator, backward difference operator, shift operator, central difference operator and mean operator.

Forward difference operator (Δ)

For the values $y_0, y_1, ..., y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, x_2, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, ..., x_n = x_0 + nh$, the forward difference operator Δ is defined on the function f(x) as,

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

That is,

$$\Delta y_i = y_{i+1} - y_i$$

Then, in particular

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)$$
$$\Rightarrow \quad \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1)$$
$$\Rightarrow \quad \Delta y_1 = y_2 - y_1$$

etc.,

$\Delta y_0, \Delta y_1, ..., \Delta y_i, ...$ are known as the **first forward differences**.

In particular,

$$\Delta^2 f(x_0) = y_2 - 2y_1 + y_0 \quad or \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0$$

In particular,

$$\Delta^3 f(x_0) = y_3 - 3y_2 + 3y_1 - y_0 \quad or \quad \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

In general the nth forward difference,

$$\Delta^n f(x_I) = \Delta^{n-1} f(x_I + h) - \Delta^{n-1} f(x_I)$$

Forward Differences Tabular Form

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0 = f(x_o)$			
		$\Delta y_0 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	$y_2 = f(x_2)$		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Example Construct the forward difference table for the following x values and its corresponding f values.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
f	0.003	0.067	0.148	0.248	0.370	0.518	0.697

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example Construct the forward difference table for the data

$$x: -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x): \quad 4 \quad 9 \quad 17 \quad 22$$

The forward difference table is as follows:

x	y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	4			
		$\Delta y_0 = 5$		
0	9		$\Delta^2 y_0 = 3$	
		$\Delta y_1 = 8$		$\Delta^3 y_0 = -6$
2	17		$\Delta^2 y_1 = -3$	
		$\Delta y_2 = 5$		
4	22			

Properties Of Forward Difference Operator

(i) Forward difference of a constant function is zero.

(ii) For the functions $f(x)$ and $g(x)$; $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$

(iii) Proceeding as in (ii), for the constants a and b,

$$\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x).$$

(iv) Forward difference of the product of two functions is given by,

$$\Delta(f(x)g(x)) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Note : Adding and subtracting $g(x+h)\,f(x)$ instead of $f(x+h)g(x)$, it can also be proved that

$$\Delta(\,f(x)g(x)) = g(x+h)\Delta f(x) + f(x)\Delta g(x)$$

(v) Forward difference of the quotient of two functions is given by

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

Higher Differences-Forward Difference Table

If n is an integer,

$$f(a+nh) = f(a) + {}^nC_1\Delta f(a) + {}^nC_2\Delta^2 f(a) + \cdots + \Delta^n f(a)$$

for the polynomial f(x) in x.

Forward Difference Table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
x_0	f_0						
x_1	f_1	Δf_0	$\Delta^2 f_0$				
				$\Delta^3 f_0$			
x_2	f_2	Δf_1	$\Delta^2 f_2$		$\Delta^4 f_0$		
				$\Delta^3 f_1$		$\Delta^5 f_0$	
x_3	f_3	Δf_2	$\Delta^2 f_2$		$\Delta^4 f_1$		$\Delta^6 f_0$
				$\Delta^3 f_2$		$\Delta^5 f_1$	
x_4	f_4	Δf_3	$\Delta^2 f_3$		$\Delta^4 f_2$		
				$\Delta^3 f_3$			
x_5	f_5	Δf_4	$\Delta^2 f_4$				
		Δf_5					
x_6	f_6						

From the forward difference table we have

$$\Delta f_0 = f_1 - f_0 \quad \text{or} \quad f_1 = f_0 + \Delta f_0$$
$$\Delta f_1 = f_2 - f_1 \quad \text{or} \quad f_2 = f_1 + \Delta f_1$$
$$\Delta f_2 = f_3 - f_2 \quad \text{or} \quad f_3 = f_2 + \Delta f_2$$

and so on. Similarly,

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 \quad \text{or} \quad \Delta f_1 = \Delta f_0 + \Delta^2 f_0$$
$$\Delta^2 f_1 = \Delta f_2 - \Delta f_1 \quad \text{or} \quad \Delta f_2 = \Delta f_1 + \Delta^2 f_1$$

and so on. Similarly, we can write

$$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 \quad \text{or} \quad \Delta^2 f_1 = \Delta^2 f_0 + \Delta^3 f_0$$
$$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1 \quad \text{or} \quad \Delta^2 f_2 = \Delta^2 f_1 + \Delta^3 f_1$$

In general,

$$\Delta^n f_0 = f_n - {}^nC_1 f_{n-1} + {}^nC_2 f_{n-2} - {}^nC_3 f_{n-3} + \ldots + (-1)^n f_0 \; .$$

$$f_n = f_0 + {}^nC_1 \Delta f_0 + {}^nC_2 \Delta^2 f_0 + \ldots + \Delta^n f_0$$

Thus

$$f_n = \sum_{i=0}^{n} {}^nC_i \, \Delta^i f_0 .$$

Result 1: The nth forward difference of a polynomial of degree n is constant when the values of the independent variable are at equal intervals.

Backward Difference Operator (∇)

For the values $y_0, y_1, ..., y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h, \ x_3 = x_0 + 3h, ..., x_n = x_0 + nh$, the **backward difference operator** ∇ is defined on the function f(x) as,

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = y_i - y_{i-1},$$

which is the **first backward difference.**

In particular, we have the first backward differences,

$$\nabla f(x_1) = y_1 - y_0; \ \nabla f(x_2) = y_2 - y_1 \ etc$$

The second backward difference is given by

$$\nabla^2 f(x_i) = \nabla(\nabla f(x_i)) = \nabla\left[f(x_i) - f(x_i - h)\right] = \nabla f(x_i) - \nabla f(x_i - h)$$
$$= \left[f(x_i) - f(x_i - h)\right] - \left[f(x_i - h) - f(x_i - 2h)\right]$$
$$= (y_i - y_{i-1}) - (y_{i-1} - y_{i-2})$$
$$= y_i - 2y_{i-1} + y_{i-2}$$

Similarly, the third backward difference, $\nabla^3 f(x_i) = y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}$ and so on.

Backward Differences Tabular Form

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_o	$y_0 = f(x_o)$			
		$\nabla y_1 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	$y_2 = f(x_2)$		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Problem: Construct the backward difference table for the data

$$x: -2 \quad 0 \quad 2 \quad 4$$

$$y = f(x): -8 \quad 3 \quad 1 \quad 12$$

Solution: The backward difference table is as follows:

x	Y=f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$
-2	-8			
		$\nabla y_1 = 3-(-8)=11$		
0	3		$\nabla^2 y_2 = -2-11 = -13$	
		$\nabla y_2 = 1-3=-2$		$\nabla^3 y_3 = 13-(-13)=26$
2	1		$\nabla^2 y_3 = 11-(2)=13$	
		$\nabla y_3 = 12-1=11$		
4	12			

Higher Differences-Backward Difference Table

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$
x_0	f_0						
x_1	f_1	∇f_1	$\nabla^2 f_2$				
x_2	f_2	∇f_2	$\nabla^2 f_3$	$\nabla^3 f_3$	$\nabla^4 f_4$	$\nabla^5 f$	
x_3	f_3	∇f_3	$\nabla^2 f_4$	$\nabla^3 f_4$	$\nabla^4 f_5$	$_5$	$\nabla^6 f_6$
x_4	f_4	∇f_4	$\nabla^2 f_5$	$\nabla^3 f_5$	$\nabla^4 f_6$	$\nabla^5 f$	
		∇f_5		$\nabla^3 f_6$		$_6$	
x_5	f_5		$\nabla^2 f_6$				
		∇f_6					
x_6	f_6						

$$\nabla f_n = f_n - f_{n-1} \quad \text{implies} \quad f_{n-1} = f_n - \nabla f_n$$

and $\nabla f_{n-1} = f_{n-1} - f_{n-2}$ implies $f_{n-2} = f_{n-1} - \nabla f_{n-1}$

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1} \quad \text{implies} \quad \nabla f_{n-1} = \nabla f_n - \nabla^2 f_n$$

From equations (1) to (3), we obtain

$$f_{n-2} = f_n - 2\nabla f_n + \nabla^2 f_n.$$

Similarly, we can show that

$$f_{n-3} = f_n - 3\nabla f_n + 3\nabla^2 f_n - \nabla^3 f_n.$$

Symbolically, these results can be rewritten as follows:

$$f_{n-1} = (1 - \nabla) f_n, \quad f_{n-2} = (1 - \nabla)^2 f_n, \quad f_{n-3} = (1 - \nabla)^3 f_n.$$

Thus, in general, we can write

$$f_{n-r} = (1 - \nabla)^r f_n.$$

i.e., $f_{n-r} = f_n - {}^r C_1 \nabla f_n + {}^r C_2 \nabla^2 f_n - \ldots + (-1)^r \nabla^r f_n$

Relation between backward difference and forward differences

1. $\Delta y_0 = y_1 - y_0 = \nabla y_1; \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0 = \nabla^2 y_2$ etc.

2. $\Delta - \nabla = \Delta \nabla$

3. $\nabla = \Delta E^{-1}$

       Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x) \quad \Rightarrow \quad \nabla = \Delta E^{-1}$$

4. $\nabla = 1 - E^{-1}$

       Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) = \left(1 - E^{-1}\right) f(x) \quad \Rightarrow \quad \nabla = 1 - E^{-1}$$

Central Differences(δ)

Central difference operator δ for a function f(x) at x_I is defined as,

$$\delta f(x_I) = f\left(x_I + \frac{h}{2}\right) - f\left(x_I - \frac{h}{2}\right), \text{ where } h \text{ being the interval of differencing.}$$

Let $y_{\frac{1}{2}} = f\left(x_0 + \frac{h}{2}\right)$. Then,

$$\delta y_{\frac{1}{2}} = \delta f\left(x_0 + \frac{h}{2}\right) = f\left(x_0 + \frac{h}{2} + \frac{h}{2}\right) - f\left(x_0 + \frac{h}{2} - \frac{h}{2}\right)$$

$$= f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0$$

$$\Rightarrow \delta y_{\frac{1}{2}} = \Delta y_0$$

Central Differences Tabular Form

x	y	δy	$\delta^2 y$	$\delta^3 y$
x_o	$y_0 = f(x_o)$			
		$\delta y_{\frac{1}{2}} = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$	
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$
x_2	$y_2 = f(x_2)$		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$	
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Higher Differences-Central Difference Table

x	f	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
x_1	f_1	$\delta f_{1/2}$	$\delta^2 f_1$		
				$\delta^3 f_{3/2}$	
x_2	f_2	$\delta f_{3/2}$	$\delta^2 f_2$		$\delta^4 f_2$
				$\delta^3 f_{5/2}$	
x_3	f_3	$\delta f_{5/2}$	$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right).$$

Thus if $f(x_i) = y_i$ then we have

$$\delta y_{\frac{1}{2}} = y_1 - y_0$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\vdots \qquad \vdots$$

$$\delta y_{\frac{n-1}{2}} = y_n - y_{n-1}$$

Differences of a Polynomial

Let us consider the polynomial of degree n in the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n ,$$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants. Let h be the interval of differencing. Then

$$f(x+h) = a_0 (x+h)^n + a_1 (x+h)^{n-1} + a_2 (x+h)^{n-2} + \ldots + a_{n-1}(x+h) + a_n$$

Now the difference of the polynomials is:

$$\Delta f(x) = f(x+h) - f(x) = a_0 \left[(x+h)^n - x^n \right] + a_1 \left[(x+h)^{n-1} - x^{n-1} \right] + \ldots \qquad + a_{n-1}(x+h-x)$$

Simplification Form

$$\Delta f(x) = a_0 n h x^{n-1} + b'\, x^{n-2} + c'\, x^{n-3} + \ldots + k'x + l'\,,$$

where b', c', $\ldots, k'$, l' are constants involving h but not x.

$$\Delta^2 f(x) = a_0 n(n-1)h^2 x^{n-2} + b''x^{n-3} + c''x^{n-4} + \ldots + q''\,.$$

$$\Delta^n f(x) = a_0 n(n-1)(n-2)(n-3) \ldots (2)(1)h^n$$

$$= a_0\,(n!)\,h^n = \text{constant}.$$

Factorial Polynomial (or) Factorial Notation

Definition

A product of the form x(x-h) (x-2h) … [x-(n-1) h] is called a factorial function and is denoted by $x^{(n)}$.

$$\therefore x^{(n)} = x(x-h)(x-2h)\cdots\left[x-(n-1)h\right].$$

Thus $x^{(1)} = x$, $x^{(2)} = x(x-1)$ and $x^{(3)} = x(x-1)$ (x-2).

We observed that $x^{(n)}$ is a polynomial of degree n with leading coefficient 1.

The following theorem shows that the formula for the first difference of $x^{(n)}$ is obtained by the simple rule of differentiation.

Theorem:

$\Delta x^{(n)} = nh\,x^{(n-1)}$. In particular when h = 1, $\Delta x^{(n)} = n\,x^{(n-1)}$.

Note: 1

From equation (1) we get the formula for first order difference, which is obtained by the simple differentiation rule.

For example, $\Delta^2 x^{(n)} = \Delta\left[nhx^{(n-1)}\right] = n\,h\,\Delta x^{(n-1)} = n(n-1)h^2\,x^{(n-2)}$ proceeding like this we get

$$\Delta^n x^{(n)} = \Delta n(n-1)(n-2)\cdots 1\,h^n\,x^0 = n!\,h^n.$$

Note: 2

Any polynomial f(x) of degree n can be expressed in the form $f(x) = c_0 x^{(n)} + c_1 x^{(n-1)} + \cdots + c_{n-1} x^{(1)} + c_n$. If f(x) is divided successively by x-0, x-1, x-2, …, x-(n-1), then the remainders give the coefficients $c_n, c_{n-1}, \cdots, c_1, c_0$.

Definition

The reciprocal factorial function $x^{(n)}$ for positive integer n is defined as

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\ldots(x+nh)}$$

As in the case of factorial function the formula for first order difference of $x^{(-n)}$ is similar to differentiation rule when $h = 1$.

Remark:

$$\Delta x^{(-n)} = (-n)h\, x^{-(n-1)}$$

$$\Delta^2 x^{(-n)} = \Delta\left[-nhx^{-(n+1)}\right]$$

$$= nh\left[-(n+1)h\right]x^{-(n+2)} = (-1)^2\, h^2\, n(n+1)x^{-(n+2)}$$

In general $\Delta^r x^{(-n)} = (-1)^r\, h^r\, n(n+1)\cdots(n+r+1)x^{-(n+1)}$.

Shift Operator(E)

Definition:

The shift operator E is defined by $Ef(x) = f(x+h)$. Hence $E^2 f(x) = Ef(x+h) = f(x+2h)$. In general, for any positive integer n

$$E^n f(x) = f(x+nh)$$

In particular we have

$$Ey_0 = y_1$$

$$E^2 y_0 = y_2$$

$$\ldots\ldots\ldots\ldots$$

$$E^n y_0 = y_n.$$

If f_0 (= y_0), f_1 (= y_1)... are the consecutive values of the function

$y = f(x)$, then we can also write

$$E f_0 = f_1 \text{ (or } E y_0 = y_1\text{)}, \quad E f_1 = f_2 \text{ (or } E y_1 = y_2\text{)}...$$

$$E^2 f_0 = f_2 \text{ (or } E^2 y_0 = y_2\text{)}, \quad E^2 f_1 = f_3 \text{ (or } E y_1 = y_3\text{)}...$$

$$E^3 f_0 = f_3 \text{ (or } E^2 y_0 = y_3\text{)}, \quad E^3 f_1 = f_4 \text{ (or } E y_1 = y_4\text{)}...$$

and so on.

Note:

$$E^m E^n f(x) = E^{m+n} f(x)$$

Example 1:

$$E\nabla = \nabla E = \Delta.$$

Solution:

$$E\nabla = E(1 - E^{-1})E = E - 1 = \Delta.$$

Also, $\nabla E = (1 - E^{-1})E = E - 1 = \Delta.$

Example 2:

$$\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)(1+\Delta)^{\frac{1}{2}} = 2 + \Delta.$$

Solution:

$$\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)(1+\Delta)^{\frac{1}{2}} = \left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)E^{\frac{1}{2}}$$

$$= E + 1$$

$$= (1 + \Delta) + 1$$

$$\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right)(1+\Delta)^{\frac{1}{2}} = 2 + \Delta.$$

The **inverse operator** E^{-1} is defined as:

$$E^{-1} f(x) = f(x - h) \qquad \ldots(3)$$

and similarly

$$E^{-n} f(x) = f(x - nh) \qquad \ldots(4)$$

Average Operator (μ)

The **average operator** μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right) \right]$$

Relations between the operators:

Operators $\Delta, \nabla, \delta, \mu$ **and** D **in terms of** E

From the definition of operators Δ and E, we have

$$\Delta f(x) = f(x+h) - f(x) = E f(x) - f(x) = (E-1) f(x).$$

Therefore,

$$\Delta = E - 1$$

From the definition of operators ∇ and E^{-1}, we have

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) = (1 - E^{-1}) f(x).$$

Therefore,

$$\nabla = 1 - E^{-1} = \frac{E-1}{E}.$$

The definition of the operators δ and E gives

$$\delta f(x) = f(x + h/2) - f(x - h/2) = E^{1/2}f(x) - E^{-1/2}f(x)$$

$$= (E^{1/2} - E^{-1/2})\, f(x).$$

The definition of the operators μ and E yields

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right] = \frac{1}{2}\left[E^{1/2} + E^{-1/2} \right] f(x).$$

Therefore,

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2} \right).$$

It is known that

$$E f(x) = f(x + h).$$

Using the Taylor series expansion, we have

$$Ef(x) = f(x) + h \, f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$= f(x) + h \, Df(x) + \frac{h^2}{2!} D^2(x) + \ldots$$

$$= \left(1 + \frac{h \, D}{1!} + \frac{h^2 D^2}{2!} + \ldots \right) f(x) = e^{hD} f(x).$$

Thus $\quad E = e^{hD}$. Or,

$$hD = \log E.$$

Simple Problems

Form the difference table for the following data.

x	50°	51°	52°	53°	54°
$y = cosx$	0.6428	0.6293	0.6157	0.6018	0.5878

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.6428				
	-0.0135			
0.6293		-0.0001		
	-0.0136		-0.0002	
0.6157		-0.0003		0.0004
	-0.0139		0.0002	
0.6018		-0.0001		
	-0.014			
0.5878				

Problem :Form the forward difference table

x	0.20	0.22	0.24	0.26	0.28	0.30
f(x)	1.6596	1.6698	1.6804	1.691	1.7024	1.7139

Solution: Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.20	1.6596					
		0.0102				
0.22	1.6698		0.0004			
		0.0106		-0.0004		
0.24	1.6804		0		0.0012	
		0.0106		0.0008		-0.0027
0.26	1.6910		0.0008		-0.0015	
		0.0114		-0.0007		
0.28	1.7024		0.0001			
		0.0115				
0.30	1.7139					

Problem: Form the backward Difference Table for the given set of values:

x	1	3	5	7	9
y	8	12	21	36	62

Solution: Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	8				
		4			
3	12		5		
		9		1	
5	21		6		4
		15		5	
7	36		11		
		26			
9	62				

Problem: Form the difference table

xᵢ	0	2	4	6	8	10	12	14
yᵢ	625	81	1	1	81	625	2401	65611

Solution: Forward, Backward and central Differences

x_i	y_i	$\Delta\, y_i$	$\Delta^2\, y_i$	$\Delta^3\, y_i$	$\Delta^4\, y_i$	$\Delta^5\, y_i$
0	625					
		-544				
2	81		464			
		-80		-384		
4	1		80		384	
		0		0		0 Forward
6	1		80		384	
		80		384		0 Central
8	81		464		384	
		544		768		0Backward
10	625		1232		384	
		1776		1152		
12	2401		2384			
		4160				
14	6561					

UNIT – III
INTERPOLATION WITH EQUAL INTERVALS

Interpolation

Interpolation

Definition

Interpolation is the process of estimating the value of a function at an intermediate point or the process of finding the value of the function inside the given range is called interpolation.

Extrapolation

Definition

Extrapolation is the process of finding the values outside the given interval.

Polynomial Interpolation

Given a set of tabular values of a function $y=f(x)$ where the explicit nature of the function is not known, then $f(x)$ is replaced by a simpler function $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree with the set of tabulated points. Any other value may be calculated from $\phi(x)$. This function $\phi(x)$ is known as interpolation function. In particular if $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called an interpolating polynomial. The existence of an interpolating polynomial is supported by Weierstras's approximation theorem which asserts that any continuous function on a closed interval can be approximated by a polynomial.

Newton's Interpolation Formula

Newton's Forward and Backward Interpolation Formula for Equal Intervals

Newton's Forward Interpolation Formula

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where $p = \dfrac{x - x_0}{h}$, h is the width of interval

$x = x_0 + ph.$

Using Newton's forward interpolation formula, find the polynomial f(x) satisfying the following data. Hence evaluate y at x = 5.

x	4	6	8	10
y	1	3	8	10

Solution

We form the difference table

x	y	Δy	Δ²y	Δ³y
4	1			
		3-1=2		
			5-2=3	
6	3			-3-3=-6
		8-3=5		
			2-5=-3	
8	8			
		10-8=2		
10	10			

There are only 4 data given. Hence the polynomial will be degree 3.

Therefore Newton's –Gregory Forward interpolation Formula is

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Here $y_0 = 1$; $p = \dfrac{x - x_0}{h} = \dfrac{x-4}{2}$

$$y = f(x) = 1 + \frac{\left(\dfrac{x-4}{2}\right)}{1!}(2) + \frac{\left(\dfrac{x-4}{2}\right)\left(\dfrac{x-4}{2}-1\right)}{2!}(3) + \frac{\left(\dfrac{x-4}{2}\right)\left(\dfrac{x-4}{2}-1\right)\left(\dfrac{x-4}{2}-2\right)}{3!}(-6)$$

$$\therefore y = f(x) = \frac{1}{8}\left[-x^3 + 21x^2 - 126x + 240\right]$$

When x= 5,

$$\therefore y = f(5) = \frac{1}{8}\left[-(5)^3 + 21(5)^2 - 126(5) + 240\right] = 1.25$$

Y = 1.25 when x = 5.

A third degree polynomial passes through the points $(0,-1)$, $(1,1)$, $(2,1)$ and $(3,-2)$ using Newton's forward interpolation formula find the polynomial. Hence find the value at 1.5.

Solution

We form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-1			
		1+1 = 2		
			0-2=-2	
1	1			-3+2=-1
		1-1=0		
			-3-0=-3	
2	1			
		-2-1=-3		
3	-2			

There are only 4 data given. Hence the polynomial will be degree 3.

Therefore Newton's –Gregory Forward interpolation Formula is

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Here $y_0 = -1$; $p = \dfrac{x-x_0}{h} = \dfrac{x-0}{1} \Rightarrow p = x$

$$y = f(x) = -1 + \frac{x}{1!} \times 2 + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$$\therefore y = f(x) = -\frac{1}{6}\left[x^3 + 3x^2 - 16x + 6\right]$$

When x= 1.5,

$$\therefore y = f(1.5) = -\frac{1}{6}\left[(1.5)^3 + 3(1.5)^2 - 16(1.5) + 6\right] = 1.3125$$

Y = 1.3125 when x = 1.5.

Newton's Backward Difference formula

Newton's backward difference formula is

$$y = f(x) = y_0 + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

where $p = \dfrac{x - x_n}{h}$, h is the width of interval

$x = x_n + ph.$

Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data: $f(-0.75) = -0.07181250$; $f(-0.5) = -0.024750$; $f(-0.25) = 0.33493750$; $f(0) = 1.10100$. Hence find $f(-1/3)$.

Solution

We form the difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.07181250			
		0.0470625		
-0.50	-0.024750		0.312625	
		0.3596875		0.09375
-0.25	0.33493750		0.400375	
		0.7660625		
0	1.10100			

Newton's backward difference formula is

$$y = f(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+3)}{3!}\nabla^3 y_3$$

where $p = \dfrac{x - x_3}{h}$; $p = 4x$.

$$\therefore y = f(x) = 1.10100 + \frac{4x}{1!}(0.7660625) + \frac{4x(4x+1)}{2!}(0.406375) + \frac{4x(4x+1)(4x+2)}{3!}$$

$$y = f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$$

$$\therefore f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101 = 0.174518518.$$

Find the missing term in the following table:

x	$y = f(x)$
0	1
1	3
2	9
3	—
4	81

Explain why the result differs from $3^3 = 27$?

Since four points are given, the given data can be approximated by a third degree polynomial in x. Hence $\Delta^4 f_0 = 0$. Substituting $\Delta = E - 1$ we get, $(E-1)^4 f_0 = 0$, which on simplification yields

$$E^4 f_0 - 4E^3 f_0 + 6E^2 f_0 - 4Ef_0 + f_0 = 0 \, .$$

Since $E^r f_0 = f_r$, the above equation becomes

$$f_4 - 4f_3 + 6f_2 - 4f_1 + f_0 = 0$$

Substituting for f_0, f_1, f_2 and f_4 in the above, we obtain

$$f_3 = 31$$

By inspection it can be seen that the tabulated function is 3^x and the exact value of $f(3)$ is 27. The error is due to the fact that the exponential function 3^x is approximated by means of a polynomial in x of degree 3.

From the data given below find the number of students whose weight is between 60 and 70.

Weight in lbs	0-40	40-60	60-80	80-100	100-120
Number of students	250	120	100	70	50

Solution: Let weight be denoted by x and

Number of students be denoted by y, i.e., $y = f(x)$.

Use Newton's forward formula to find y where x lies between $60 - 70$.

Newton's forward formula is

$$y(x) = f(x) = f(x_0 + uh)$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

We rewrite the table as cumulative table showing the number of students less than x lbs.

x	Below 40	Below 60	Below 80	Below 100	Below 120
y	250	370	470	540	590

Newton's forward difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		−20		
		120		−10	
Below 80	470		−30		20
		70		10	
Below 100	540		−20		
		50			
Below 120	590				

Here $u = \dfrac{x - x_0}{h} = \dfrac{x - 40}{20}$.

When $x = 70$, $u = \dfrac{70 - 40}{20} = 1.5$

$$y(x = 70) = 250 + \dfrac{1.5}{1!}(120) + \dfrac{1.5(1.5 - 1)}{2!}(-20) + \dfrac{1.5(1.5 - 1)(1.5 - 2)}{3!}(-10)$$
$$+ \dfrac{1.5(1.5 - 1)(1.5 - 2)(1.5 - 3)}{4!}(20)$$
$$= 423.59 = 424$$

$\therefore$ Number of students whose weight is $70 = 424$

$\therefore$ Number of students whose weight is between $60 - 70 = 424 - 370 = 54$.

Central Difference Interpolation

Newton's forward and backward interpolation formula are applicable for interpolation near the beginning and near the end of the tabulated arguments, respectively.

Interpolation near the centre of the tabulated arguments. For this purpose we use central difference interpolation formula. Gauss's forward interpolation formula, Gauss's backward interpolation formula, Sterling's formula, Bessel's formula, Laplace-Everett's formula are some of the various central difference interpolation formulas.

Let us consider some equidistant arguments with interval of difference, say; h and corresponding function values are given. Let x_0, be the central point among the arguments.

For interpolation at the point x near the central value, let $f(x_0) = y_0$, $f(x_0 - h) = y_{-1}$, $f(x_0 + h) = y_1$, $f(x_0 - 2h) = y_{-2}$, $f(x_0 + 2h) = y_2$, $f(x_0 - 3h) = y_{-3}$, $f(x_0 + 3h) = y_3$ and so on.

For the values $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3$ the forward difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$x_0 - 3h$	y_{-3}						
		Δy_{-3}					
$x_0 - 2h$	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
$x_0 - h$		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
x_0		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
$x_0 + h$		Δy_1		$\Delta^3 y_0$			
	y_2		$\Delta^2 y_1$				
$x_0 + 2h$		Δy_2					
	y_3						
$x_0 + 3h$							

Gauss' Central Difference Formulae

We consider two central difference formulae.

(i) Gauss's forward formula

We consider the following table in which the central coordinate is taken for convenience as y_0 corresponding to $x = x_0$

Gauss's Forward formula is

$$f_p = f_0 + G_1 \Delta f_0 + G_2 \Delta^2 f_{-1} + G_3 \Delta^3 f_{-1} + G_4 \Delta^4 f_{-2} + ...,$$

where $G_1, G_2, ...$ are given by

$$G_1 = p$$

$$G_2 = \frac{p(p-1)}{2!}$$

$$G_3 = \frac{(p+1)p(p-1)}{3!},$$

$$G_4 = \frac{(p+1)p(p-1)(p-2)}{4!},$$

Table: Gauss' Forward Formula

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy^2					
x_3	y_3						

(ii) Gauss Backward Formula

Gauss backward formula is

$$f_p = f_0 + G_1' \Delta f_{-1} + G_2' \Delta f_{-1} + G_3' \Delta f_{-2} + G_4' \Delta^4 f_{-2} + \ldots$$

where $G_1', G_2', \ldots$ are given by

$$G_1' = p,$$

$$G_2' = \frac{p(p+1)}{2!},$$

$$G_3' = \frac{(p+1)p(p-1)}{3!},$$

$$G_4' = \frac{(p+2)(p+1)p(p-1)}{4!},$$

Gauss Forward Difference formula

y_p =

$$\left[y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-2}) + \ldots \ldots \right]$$

This version of Newton's Forward Interpolation Formula is known as the Gauss's Forward Interpolation Formula.

We observe that the formula contains y_0 and the even differences $\Delta^2 y_{-1,}$

$\Delta^4 y_{-2,}$ ..... which lie on the line containing x_0 (called the central line) and the odd differences $\Delta y_0, \Delta^3 y_{-1}$ ...... which lie on the line just below this line, in the difference table.

Gauss Backward Difference Formula

$$y_p =$$

$$\left[y_0 + p(\Delta y_{-1}) + \frac{p(p+1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p-1)(p+1)}{3!}(\Delta^3 y_{-2}) + \frac{p(p-1)(p+1)(p-2)}{4!}(\Delta^4 y_{-2}) + \ldots \ldots \right]$$

The version of the Newton's Forward Interpolation Formula is known as the Gauss's Backward Interpolation Formula.

Observe the formula contains y0 and the even differences $\Delta^2 y_{-1}$, $\Delta^4 y_{-2}$ ....

which lie on the central line and the odd differences Δy_{-1}, $\Delta^3 y_{-2}$... which lie on the line just above this line.

Example From the following table, find the value of $e^{1.17}$ using Gauss' forward formula.

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Solution

Here we take $x_0 = 1.15$, $h = 0.05$.

Also, $x_p = x_0 + ph$

$$1.17 = 1.15 + p(0.05),$$

which gives

$$p = \frac{0.02}{0.05} = \frac{1}{4}$$

The difference table is given below:

x	e^x	Δ	Δ^2	Δ^3	Δ^4
1.00	2.7183				
		0.1394			
1.05	2.8577		0.0071		
		0.1465		0.0004	
1.10	3.0042		0.0075		0
		0.1540		0.0004	
1.15	3.1582		0.0079		0
		0.1619		0.0004	
1.20	3.3201		0.0083		0.0001
		0.1702		0.0005	
1.25	3.4903		0.0088		
		0.1790			
1.30	3.6693				

Using Gauss's forward difference formula we obtain

$$e^{1.17} = 3.1582 + \frac{2}{5}(0.1619) + \frac{(2/5)(2/5-1)}{2}(0.0079)$$

$$+ \frac{(2/5+1)(2/5)(2/5-1)}{6}(0.0004)$$

$$= 3.1582 + 0.0648 - 0.0009 = 3.2221.$$

Stirling's Formula

$$y(x) = y(x_0 + uh) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Using Stirling's formula obtain the value of f(12) given

x	5	10	15	20
f(x)	54.14	60.54	67.72	75.88

Solution: Difference Table

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1}	5	54.14			
			6.4		
x_0	10	60.54		0.78	
			7.18		0.2
x_1	15	67.72		0.98	
			8.16		
x_2	20	75.88			

$$u = \frac{x - x_0}{h} = \frac{12 - 10}{5} = 0.4$$

Using Stirling's formula,

$$y(x) = y(x_0 + uh) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$
$$\frac{u^2(u^2 - 1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

$$y(12) = 60.54 + (0.4)\frac{(7.18 + 6.4)}{2} + \frac{(0.4)^2}{2}(0.78)$$

$$y(12) = 63.318$$

Bessel's Formula

$$y(x) = y(x_0 + uh) = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \dots$$

Obtain the value of y(5), using Bessel's formula given

x	0	4	8	12
f(x)	143	158	177	199

Solution: **Difference Table**

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1}	0	143			
			15		
x_0	4	158		4	
			19		-1
x_1	8	177		3	
			22		
x_2	12	199			

$$u = \frac{x - x_0}{h} = \frac{5 - 4}{4} = 0.25$$

Using Bessel's formula,

$$y(x) = y(x_0 + uh) = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

We have,

$$y(5) = \frac{158 + 177}{2} + (0.25 - 0.5)(19) + \frac{(0.25)(0.25 - 1)}{2}\left(\frac{4 + 3}{2}\right)$$

$$+ \frac{\left(0.25 - \frac{1}{2}\right)(0.25)(0.25 - 1)}{6}(-1)$$

$$= 167.5 - 4.75 - 0.3281 + 0.00781$$

$$= 162.42$$

$$\boxed{y(5) = 162.42}$$

UNIT – IV
INTERPOLATION WITH UNEQUAL INTERVALS

Newton's Divided Difference Formula

Let the function $y = f(x)$ takes the values $f(x_0)$, $f(x_1)$, ..., $f(x_n)$ corresponding to the values x_1-x_0, $x_2 - x_1$, $x_3 - x_2$, ..., x_n-x_{n-1} need not to be equal.

Be first divided difference of $f(x)$ for the argument x_0, x_1 is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

it is denoted by $f(x_0, x_1)$ or $[x_0, x_1]$ or $\Delta f(x_0)$.

(i.e) $f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

First divided difference for arguments x_0, x_1 :

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1) = \underset{x_1}{\triangle} f(x_0) = [x_0, x_1](or)[x_1, x_0] = \underset{x_0}{\triangle} f(x_1)$$

First divided difference for arguments x_1, x_2 :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2) = \underset{x_2}{\triangle} f(x_1) = [x_1, x_2](or)[x_2, x_1] = \underset{x_1}{\triangle} f(x_2)$$

Second divided difference for arguments x_1, x_2, x_3 :

$$\frac{\underset{x_3}{\triangle} f(x_2) - \underset{x_2}{\triangle} f(x_1)}{x_3 - x_1} = f(x_1, x_2, x_3) = \underset{x_3 x_2}{\triangle^2} f(x_1) = [x_1, x_2, x_3]$$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$ $= \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$		
x_1	$f(x_1)$			
			$f(x_0, x_1, x_2)$ $= \dfrac{f(x_2 x_1) - f(x_1 x_0)}{x_2 - x_0}$	
		$f(x_1, x_2)$ $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$		$f(x_0, x_1, x_2, x_3)$ $= \dfrac{f(x_1 x_2 x_3) - f(x_0 x_1 x_2)}{x_3 - x_0}$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$ $= \dfrac{f(x_3 x_2) - f(x_2 x_1)}{x_3 - x_1}$	
				$f(x_1, x_2, x_3, x_4)$ $= \dfrac{f(x_2 x_3 x_4) - f(x_1 x_2 x_0)}{x_4 - x_1}$
		$f(x_2, x_3)$ $= \dfrac{f(x_2) - f(x_3)}{x_3 - x_2}$	$f(x_2, x_3, x_4)$ $= \dfrac{f(x_4 x_3) - f(x_3 x_2)}{x_4 - x_1}$	
x_3	$f(x_3)$			
		$f(x_0, x_1)$ $= \dfrac{f(x_1) - f(x_0)}{x_4 - x_3}$		
x_4	$f(x_4)$			

Properties of Divided Differences

Property: 1

The value of any divided difference is independent of the order of the arguments. That is, the divided differences are symmetrical in all their arguments.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0) \tag{1}$$

Again,

$$f(x_0, x_1) = \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \tag{2}$$

In the same way,

$$f(x_1, x_0) = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \tag{3}$$

From equation (2) and equation (3), we have $f(x_0, x_1) = f(x_1, x_0)$.

Similarly,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0} \left[\left(\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1} \right) - \left(\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \right) \right]$$

$$= \frac{1}{x_2 - x_0} \left[\left(\frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0} \right) f(x_1) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} \right]$$

$$= \frac{1}{x_2 - x_0} \left[\frac{x_2 - x_0}{(x_1 - x_2)(x_1 - x_0)} f(x_1) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} \right]$$

$$f(x_0, x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \qquad (4)$$

From equation (4), we find

$$f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_1, x_2, x_0) = \cdots$$

This shows that $f(x_0, x_1, x_2)$ is independent of the order of the arguments.

By mathematical induction, we can prove that

$$f(x_0, x_1, x_2, \cdots, x_n) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} + \cdots$$

$$+ \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}.$$

This is symmetrical we know that any two arguments. Therefore, the divided differences are symmetrical we know that any two arguments.

Property: 2

The operator Δ is linear.

Proof:

If $f(x)$ and $g(x)$ are two functions α and β are constants, then

$$\Delta[\alpha f(x) + \beta g(x)] = \frac{[\alpha f(x_1) + \beta g(x_1)][\alpha f(x_0) + \beta g(x_0)]}{x_1 - x_0}$$

$$= \alpha \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \beta \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$\Delta[\alpha f(x) + \beta g(x)] = \alpha \Delta f(x) + \beta \Delta g(x).$$

Corollary: 1

Setting $\alpha = \beta = 1$, $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$.

Corollary: 2

Setting $\beta = 0$, $\Delta[\alpha f(x)] = \alpha \Delta f(x)$.

Property: 3

The nth divided differences of a polynomial of degree n are constants.

Proof:

Taking $f(x) = x^n$ where n is a positive integer,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^n - x_0^n}{x_1 - x_0}$$

$$= x_1^{n-1} + x_0\, x_1^{n-2} + x_0^2\, x_1^{n-3} + \cdots + x_0^{n-1}$$

= a polynomial function of degree (n-1) and symmetrical in x_0, x_1 with leading coefficient 1.

Again,

$$f(x_0, x_1 x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{\left(x_2^{n-1} + x_1 x_2^{n-2} + \cdots + x_1^{n-1}\right) - \left(x_0^{n-1} + x_1 x_0^{n-2} + \cdots + x_1^{n-1}\right)}{x_2 - x_0}$$

$$= \frac{x_2^{n-1} - x_0^{n-1}}{x_2 - x_0} + \frac{x_1\left(x_2^{n-2} - x_0^{n-2}\right)}{x_2 - x_0} + \cdots + \frac{x_1^{n-2}\left(x_2 - x_0\right)}{x_2 - x_0}$$

$$= \left(x_2^{n-2} + x_0 x_2^{n-3} + \cdots + x_0^{n-2}\right) + x_1\left[x_2^{n-3} + x_0 x_2^{n-4} + x_0^{n-3}\right] + \cdots + x_1^{n-2}$$

= a polynomial of degree (n - 2) and symmetrical i x_0, x_1, x_2 with leading
    coefficient 1.

Proceeding in this way, the rth divided differences of x^n will be a polynomial of degree $(n-r)$ and symmetrical in $x_0, x_1, x_2, ..., x_r$ with leading coefficient 1.

Hence n^{th} order divided differences of x^n will be a polynomial of degree $n - n = 0$, with leading coefficient 1. That is, its value is 1.

That is $\Delta^n x^n = 1$.

$\Delta^{n+i} x^n = 0$, for $i = 1, 2, ...$

Hence, $\Delta^n \left[a_0 x^n + a_1 x^{n-1} + \cdots a_n \right]$

$$= a_0 \Delta^n x^x + a_1 \Delta^n x^{n-1} + \cdots + \Delta^n a_n$$

$$= a_0 \ 1 + 0 + 0 + \cdots + 0 = a_0.$$

Note:

Conversely, if the nth divided difference of a polynomial is constant, then the polynomial is degree of n.

Relation between Divided Differences and Forward Differences

If the arguments x_0, x_1, x_2, ... are equally spaced, then we have,

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots x_n - x_{n-1} = h.$$

$$\Delta f(x_0) = f(x_1, x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}$$

$$\Delta^2 f(x_0) = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_0} = \frac{\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)}{2^h}$$

$$= \frac{1}{2h^2}\Delta^2 f(x_0)$$

Similarly,

$$\Delta^3 f(x_0) = \frac{\Delta^3 f(x_0)}{3!h^3}$$

$$\Delta^n f(x_0) = \frac{\Delta^n f(x_0)}{n!h^n}.$$

Newton's divided difference interpolation formula for unequal intervals

$$y = f(x) = f(x_0) + (x = x_0)f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3) + \cdots$$

$$+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})f(x_0, x_1, x_2 x_3, \cdots x_n).$$

Problem

Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ and $f(15)$ given the following table:

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Solution

We form the divided difference table since the intervals are unequal.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\dfrac{100-48}{5-4}=52$			
5	100		$\dfrac{97-52}{7-4}=15$		
		$\dfrac{294-100}{7-5}=97$		$\dfrac{21-15}{10-4}=1$	
7	294		$\dfrac{202-97}{10-5}=21$		0
		$\dfrac{900-294}{10-7}=202$		$\dfrac{27-21}{11-5}=1$	
10	900		$\dfrac{310-202}{11-7}=27$		0
		$\dfrac{1210-900}{11-10}=310$		$\dfrac{33-27}{13-7}=1$	
11	1210		$\dfrac{409-310}{13-10}=33$		
		$\dfrac{2028-1210}{13-11}=409$			
13	2028				

By Newton's Divided Difference interpolation formula is

$$y = f(x) = f(x_0) + (x = x_0)f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3)$$

Here $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 13$

Also, $f(x_0) = 48$, $f(x_0, x_1) = 52$, $f(x_0, x_1, x_2) = 15$, $f(x_0, x_1, x_2 x_3) = 1$

$y = f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7).1$

$f(2) = 48 + (2-4)(52) + (2-4)(2-5)(15) + (2-4)(2-5)(2-7)$

$f(2) = 4$

$f(8) = 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)$

$f(8) = 448$

$f(15) = 48 + 11 (52) + (11)(10)(15) + (11)(10)(8)$

$f(15) = 3150.$

Using Newton's divided difference formula, find u(3) given that u(1) = -26, u(2) = 12, u(4) = 256, u(6) = 844.

Solution

We form the divided difference table since the intervals are unequal.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	-26			
2	12	$\dfrac{12+26}{2-1} = 38$	$\dfrac{122-38}{4-1} = 28$	$\dfrac{43-28}{6-1} = 3$
4	256	$\dfrac{256-12}{4-2} = 122$	$\dfrac{294-122}{6-2} = 43$	
6	844	$\dfrac{844-256}{6-4} = 294$		

Newton divided difference interpolation formula is

$$y = f(x) = f(x_0) + (x = x_0)f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3)$$

Here,

$$y = u(x) = u(x_0) + (x - x_0) u(x_0, x_1) + (x - x_0)(x - x_1) u(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2) u(x_0, x_1, x_2, x_3)$$

Here,

$$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 6$$

$$u(x_0) = -26, \quad u(x_0,x_1) = 38, \quad u(x_0,x_1,x_2) = 28, \quad u(x_0,x_1,x_2,x_3) = 3$$

$$u(x) = -26 + (x-1)(38) + (x-1)(x-2)(28) + (x-1)(x-2)(x-3)(3)$$

for x = 3,

$$y = u(x) = -26 + (3-1)(38) + (3-1)(3-2)(28) + (3-1)(3-2)(3-4)(3)$$

$$= -26 + 76 + 56 - 6$$

$$u(3) = 100.$$

Problem: Obtain Newton's divided difference interpolating polynomial satisfied by $(-4,1245)$, $(-1,33)$, $(0,5)$, $(2,9)$ *and* $(5,1335)$.

Solution: Newton's divided difference interpolating polynomial is given by,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + \ldots +$$
$$(x - x_0)(x - x_1)\ldots(x - x_{n-1})f(x_0, x_1, \ldots, x_n)$$

Here x values are gives as, -4, -1, 0, 2 and 9. Corresponding f(x) values are 1245,33,5,9 and 1335.

Hence the divided difference as shown in the following table:

X	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
-4				
	-404			
-1		**94**		
	-28		**-14**	
0		10		3
	2		13	
2		88		
	442			
5				

Given $f(x_0) = 1245$. From the table, we can observe that

$$f(x_0, x_1) = -404; \quad f(x_0, x_1, x_2) = 94;$$
$$f(x_0, x_1, x_2, x_3) = -14 \ and \ f(x_0, x_1, x_2, x_3, x_4) = 3$$

Hence the interpolating polynomial is,

$$f(x) = 1245 + (x - (-4)) \times (-404) + (x - (-4))(x - (-1)) \times 94$$
$$+ (x - (-4))(x - (-1))(x - 0) \times 14 + (x - (-4))(x - (-1))(x - 0)(x - 2) \times 3$$

$$\Rightarrow f(x) = 1245 - 404(x + 4) + 94(x + 4)(x + 1)$$
$$+ 14(x + 4)(x + 1)(x - 0) + 3(x + 4)(x + 1)(x - 0)(x - 2)$$

On simplification, we get,

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Lagrange's interpolation formula for unequal intervals

Let $y=f(x)$ be the function such that $f(x)$ is taking the values $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$.

In the case of the values of independent variable are not equally spaced and when the differences of dependent variable are not small, we will use Lagrange's interpolation formula.

Let $f(x)$ be a polynomial in x of degree n. Lagrange's interpolation formula for unequal intervals is

$$y = f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_1)\cdots(x_0 - x_n)}f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)(x - x_3)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}f(x_1) + \cdots +$$

$$\frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})}f(x_n).$$

Using Lagrange's interpolation formula, find the value corresponding to x=10 from the following table:

x	5	6	9	11
y	12	13	14	16

Solution

Given $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $x = 10$

$Y_0 = f(x_0) = 12$

$Y_1 = f(x_1) = 13$

$Y_2 = f(x_2) = 14$

$Y_3 = f(x_3) = 16$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} (12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} (13) +$$

$$\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} (14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} (16)$$

$$= \frac{4.1.1}{1.4.6} (12) + \frac{5.1.(-1)}{1.3.5} (13) + \frac{5.4.1}{4.3.2} (14) + \frac{5.4.1}{4.3.2} (14) + \frac{5.4.1}{6.5.2} (16)$$

$$= 14.63.$$

$$Y = f(x) = f(10) = 14.63.$$

Find the second degree polynomial fitting the following data:

x	1	2	4
y	4	5	13

Solution: Here $x_0 = 1, x_1 = 2, x_2 = 4$

$$y_0 = 4, y_2 = 5, y_2 = 13$$

By Lagrange's formula for three points is

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$y = \frac{(x^2 - 6x + 8)}{(-1)(-3)}(4) + \frac{(x^2 - 5x + 4)}{(1)(-2)}(5) + \frac{(x^2 - 3x + 2)}{(3)(2)}(13)$$

$$= \frac{(x^2 - 6x + 8)}{3}(4) + \frac{(x^2 - 5x + 4)}{-2}(5) + \frac{(x^2 - 3x + 2)}{6}(13)$$

$$= \frac{1}{6}\left[8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26\right]$$

$$= \frac{1}{6}\left[6x^2 - 12x + 30\right]$$

$$y = f(x) = x^2 - 2x + 5$$

Inverse interpolation by Lagrange's method

The process of finding a value of x for the corresponding value of y is called inverse interpolation. In such a case, we will take y as independent variable and x as dependent variable.

Therefore the Lagrange's inverse interpolation formula is

$$x = f(y) = \frac{(y - y_1)(y - y_2)\cdots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\cdots(y_0 - y_n)}.f(y_0) +$$

$$\frac{(y - y_0)(y - y_2)\cdots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\cdots(y_1 - y_n)}.f(y_1) + \cdots +$$

$$\frac{(y - y_0)(y - y_1)(y - y_2)\cdots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)(y_n - y_2)\cdots(y_n - y_{n-1})}.f(y_n).$$

Find the value of x, corresponding to y = 100 from the following table:

x	3	5	7	9	11
y	6	24	58	108	174

Solution

Given $y_0 = 6$, $y_1 = 24$, $y_2 = 58$, $y_3 = 108$, $y_4 = 174$ also, y = 100

$x = f(y)$;

$x_0 = f(y_0) = 3$, $x_1 = f(y_1) = 5$, $x_2 = f(y_2) = 7$, $x_3 = f(y_3) = 9$

The Lagrange's formula for inverse interpolation is

$$x = f(y) = \frac{(y - y_1)(y - y_2)\cdots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\cdots(y_0 - y_n)}.f(y_0) +$$

$$\frac{(y - y_0)(y - y_2)\cdots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\cdots(y_1 - y_n)}.f(y_1) + \cdots +$$

$$\frac{(y - y_0)(y - y_1)(y - y_2)\cdots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)(y_n - y_2)\cdots(y_n - y_{n-1})} \cdot f(y_n)$$

$$= 0.3534 - 1.5155 + 2.8870 + 7.0676 - 0.1369$$

$$x = f(y) = 8.6556.$$

UNIT –V
NUMERICAL DIFFERENTIATION
&
INTEGRATION

Numerical differentiation

The problem of **numerical differentiation** is the determination of approximate value of the derivative of a function f at a given point.

- **Derivative using Newton's Forward difference Formula**

For finding the derivative at a point near to the beginning of the tabular values, Newton's Forward difference Formula is used. For the values $y_0, y_1, ..., y_n$ of a function y=f(x), corresponding to the equidistant values $x_0, x_1, x_2, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, ..., x_n = x_0 + nh$, Newton's Forward difference Formula is,

$$f(x) = f(x_0 + uh) = y_0 + u[\Delta y_0] + \frac{u(u-1)}{2!}[\Delta^2 y_0]$$
$$+ \frac{u(u-1)(u-2)}{3!}[\Delta^3 y_0] + ... + \frac{u(u-1)(u-2)....(u-n+1)}{n!}[\Delta^n y_0]$$

where, $u = \frac{x - x_0}{h}$.

The derivative of $f(x)$ with respect to x, where x is any point in the interval $[x_0, x_n]$ is obtained as follows:

$$\frac{d}{dx}f(x) = \frac{d}{du}f(x) \times \frac{du}{dx}, \text{ by chain rule}$$

$$= \frac{d}{du}f(x) \times \frac{d}{dx}\left(\frac{(x-x_0)}{h}\right) = \frac{d}{du}f(x) \times \frac{1}{h}$$

$$\Rightarrow \quad \frac{d}{dx}f(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\left[\Delta^2 y_0\right] + \frac{3u^2-6u+2}{3!}\left[\Delta^3 y_0\right] + \frac{4u^3-18u^2+22u-6}{24}\left[\Delta^4 y_0\right] + \ldots\right]$$

When $x = x_0$, we get u=0. Thus,

$$\frac{d}{dx}f(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{2}{6}\Delta^3 y_0 - \frac{6}{24}\Delta^4 y_0 + \ldots\right]$$

The second derivative of $f(x)$ is

$$\frac{d^2}{dx^2}f(x) = \frac{d}{du}\left(\frac{d}{dx}f(x)\right) \times \frac{du}{dx}$$

$$= \frac{d}{du}\left(\frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\left[\Delta^2 y_0\right] + \frac{3u^2-6u+2}{3!}\left[\Delta^3 y_0\right] + \frac{4u^3-18u^2+22u-6}{24}\left[\Delta^4 y_0\right] + \ldots\right]\right) \times \frac{1}{h}$$

$$= \frac{1}{h^2}\left[\frac{2}{2!}\left[\Delta^2 y_0\right] + \frac{6u-6}{3!}\left[\Delta^3 y_0\right] + \frac{12u^2-36u+22}{24}\left[\Delta^4 y_0\right] + \ldots\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\left[\Delta^3 y_0\right] + \frac{6u^2 - 18u + 11}{12}\left[\Delta^4 y_0\right] + \ldots\right]$$

In similar way,

$$\frac{d^3}{dx^3} f(x) = \frac{d}{du}\left[\frac{d^2}{dx^2} f(x)\right] \times \frac{du}{dx} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u - 18}{12}\left[\Delta^4 y_0\right] + \ldots\right]$$

When $x = x_0$, and $u = 0$, we have

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right] \quad \text{and}$$

$$\frac{d^3}{dx^3} f(x) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

- **Derivative using Newton's Backward difference Formula**

To find the derivative at a point near to the end of the tabular values, Newton's backward difference Formula is used. For the equidistant arguments, Newton's backward difference Formula is,

$$f(x) = f(x_n + uh) = y_n + u[\nabla y_n] + \frac{u(u+1)}{2!}[\nabla^2 y_n]$$

$$+ \frac{u(u+1)(u+2)}{3!}[\nabla^3 y_n] + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!}[\nabla^n y_n]$$

where $u = \dfrac{x - x_n}{h}$

$$\frac{d}{dx} f(x) = \frac{d}{du} f(x) \times \frac{du}{dx}$$

$$= \frac{d}{du} f(x) \times \frac{d}{dx}\left(\frac{(x - x_n)}{h}\right) = \frac{d}{du} f(x) \times \frac{1}{h}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}[\nabla^2 y_n] + \frac{3u^2 + 6u + 2}{3!}[\nabla^3 y_n] + \frac{4u^3 + 18u^2 + 22u + 6}{24}[\nabla^4 y_n] + \dots\right]$$

$$\frac{d^2}{dx^2}f(x) = \frac{d}{du}\left[\frac{d}{dx}f(x)\right] \times \frac{du}{dx} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\left[\nabla^3 y_n\right] + \frac{6u^2+18u+11}{12}\left[\nabla^4 y_n\right] + ...\right], \text{ and}$$

$$\frac{d^3}{dx^3}f(x) = \frac{d}{du}\left[\frac{d^2}{dx^2}f(x)\right] \times \frac{du}{dx} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12u+18}{12}\nabla^4 y_n + ...\right]$$

At $x = x_n$, $u = 0$. The above gives,

$$\frac{d}{dx}f(x) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...\right]$$

$$\frac{d^2}{dx^2}f(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right] \text{ and}$$

$$\frac{d^3}{dx^3}f(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right].$$

The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds.

Time(sec)	0	5	10	15	20
Velocity (m/sec)	0	3	14	69	228

Find (a) Initial acceleration using the entire data   (b) Final acceleration.

Solution: The difference table is

Time $t = x$	Velocity $v = y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
$0 (= x_0)$	$0 (= y_0)$				
		$3 (= \Delta y_0)$			
5	3		$8 \left(= \Delta^2 y_0\right)$		
		11		$36 \left(= \Delta^3 y_0\right)$	
10	14		44		$24 \left(= \Delta^4 y_0 \text{ or } \nabla^4 y_n\right)$
		55		$60 \left(= \nabla^3 y_n\right)$	
15	69		$104 \left(= \nabla^2 y_n\right)$		
		$159 (= \nabla y_n)$			
20	$228 (= y_n)$				

Here $x_0 = 0, h =$ interval length $= 5$

(a) WKT, acceleration $= \dfrac{dv}{dt} =$ rate of change of velocity

To find initial acceleration, put $\left(\dfrac{dv}{dt}\right)_{t=t_0} = \left(\dfrac{dv}{dt}\right)_{t=0}$.

i.e., initial acceleration exists at $t = 0 = x_0$ [which is nearer to beginning of the table], so we use

Newton's forward difference formula for first derivative.

$\therefore$ Newton's forward difference interpolation formula is

$$y(x) = f(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Now, Newton's forward difference formula for first derivative at $x = x_0 = 0$ [$\Rightarrow u = 0$]

i.e., $y'(x_0) = \left(\dfrac{dv}{dt}\right)_{t=0} = \dfrac{1}{h}\left[\Delta y_0 - \dfrac{1}{2}\Delta^2 y_0 + \dfrac{1}{3}\Delta^3 y_0 - \dfrac{1}{4}\Delta^4 y_0 + \cdots\right]$

$= \dfrac{1}{5}\left[3 - \dfrac{1}{2}(8) + \dfrac{1}{3}(36) - \dfrac{1}{4}(24)\right] = 1$

$\Rightarrow y'(0) = 1$

$\therefore$ Initial acceleration (acceleration when $t = 0$) is 1 m/sec^2.

(b) Final acceleration exists at $t = 20 = x_4$ [Nearer to ending of table], so use Newton's backward difference interpolation formula for first derivative, put $\left(\dfrac{dv}{dt}\right)_{t=t_n} = \left(\dfrac{dv}{dt}\right)_{t=20}$.

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{5}\left[159 + \frac{1}{2}(104) + \frac{2}{6}(60) + \frac{6}{24}(24)\right]$$

$$= \frac{1}{5}(237) = 47.2 \text{ m / sec}^2$$

Problem: Compute $f''(0)$ and $f'(0.2)$ from the following tabular data.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	1.16	3.56	13.96	41.96	101.00

Solution:

Since $x = 0.0$ and 0.2 appear at and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The forward difference table for the given data is:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.0	1.00					
		0.16				
0.2	1.16		2.24			
		2.40		5.76		
0.4	3.56		8.00		3.84	
		10.40		9.60		0.00
0.6	13.96		17.60		3.84	
		28.00		13.44		
0.8	41.96		31.04			
		59.04				
1.0	101.00					

Here $x_0 = 0$, and h=0.2. At $x = 0$, $u = \dfrac{(x - x_0)}{h} = 0$,

The second derivative at $x = 0$ is given by Newton's forward formula:

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \ldots \right]$$

$$f''(0) = \frac{1}{(0.2)^2} \left[2.24 - 5.76 + \frac{11}{24}(3.84) - \frac{5}{6}(0) \right] = 0 .$$

For x=0.2, $u = \dfrac{(0.2 - 0.0)}{0.2} = 1$.

By Newton's forward formula, we have the derivative of f(x) at a point x is,

$$\frac{d}{dx} f(x) = \frac{1}{h} \left[\Delta y_0 + \frac{2u - 1}{2!} \left[\Delta^2 y_0 \right] + \frac{3u^2 - 6u + 2}{3!} \left[\Delta^3 y_0 \right] + \frac{4u^3 - 18u^2 + 22u - 6}{24} \left[\Delta^4 y_0 \right] + \ldots \right]$$

Hence,

$$\frac{d}{dx} f(x) \bigg|_{x=0.2} = \frac{1}{0.2} \left[0.16 + \frac{2 \times 1 - 1}{2!} [2.24] + \frac{3 \times 1^2 - 6 \times 1 + 2}{3!} [5.76] + \frac{4 \times 1^3 - 18 \times 1^2 + 22 \times 1 - 6}{24} [3.84] \right]$$

= 3.2, on simplification.

If the arguments are not equidistant, the approximating polynomial for the given tabular points is found by Newton's divided difference formula or Lagrange's interpolation formula. Then the derivative of the function can get at any x in the range.

For example: We find the first derivative of a function at 0, using the points $(-4, 1245), (-1, 33), (0, 5), (2, 9)$ *and* $(5, 1335)$ where x values are not equidistant. We can get the approximating polynomial by Newton's divided difference formula.

The table of divided differences is,

x	y	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
-4	1245				
		-404			
-1	33				
		-28	94		
0	5			-14	
		2			3
2	9		10		
		442		13	
5	1335				
			88		

Given $f(x_0) = 1245$. From the table, we can observe that

$$f(x_0, x_1) = -404; \quad f(x_0, x_1, x_2) = 94;$$
$$f(x_0, x_1, x_2, x_3) = -14 \quad and \quad f(x_0, x_1, x_2, x_3, x_4) = 3$$

Hence the interpolating polynomial is

$$f(x) = 1245 + (x - (-4)) \times (-404) + (x - (-4))(x - (-1)) \times 94$$
$$+ (x - (-4))(x - (-1))(x - 0) \times (-14) + (x - (-4))(x - (-1))(x - 0)(x - 2) \times 3$$

On simplification, we get

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 .$$

Then,

$$f'(x) = 12x^3 - 15x^2 + 12x - 14$$

Hence,
$$f'(0) = -14 .$$

Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots\right] \qquad (1)$$

substitute $h, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{ an equation in } u \qquad (2)$$

For maxima or minima is obtained by the equation by

$$\frac{dy}{dx} = 0 \text{ (or) } y'(x) = 0$$

Put RHS of (1) = 0, find $u = u_1, u_2, \cdots$

Find $\dfrac{d^2y}{dx^2}$ from (2):

Suppose $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } u_1} = -\text{ve} \Rightarrow u_1$ is maximum point

$\Rightarrow y(u_1) = $ maximum value of y

Suppose $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } u_2} = +\text{ve} \Rightarrow u_2$ is minimum point

$\Rightarrow y(u_2) = $ minimum value of y

Find the value of $f'(8), f''(9)$, maximum and minimum value from the following data, using an approximate interpolation formula.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Solution: The values of x are unequally spaced.

To find $f(x)$, we use Newton's divided difference formula (or) Lagrange formula.

WKT, Newton's Divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \cdots \tag{1}$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		
		310			
11	1210				

$$\therefore (1) \Rightarrow f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$$

$$= 48 + 52x - 208 + 15\left[x^2 - 9x + 20\right] + x^3 - x^2(16) + x(83) - 140$$

$$= x^3 + x^2(-16 + 15) + x(83 - 135 + 52) - 208 + 300 - 140 + 48$$

$$= x^3 - x^2 + x(0) + 0$$

$$\therefore f(x) = x^3 - x^2$$

$$f'(x) = 3x^2 - 2x, \quad f'(8) = 176$$

$$f''(x) = 6x - 2, \quad f''(9) = 52$$

To find maximum and minimum

$$\text{Put } f'(x) = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, x = \frac{2}{3}$$

At $x = 0$, $f''(x = 0) = 6(0) - 2 = -2 < 0$

$\Rightarrow x = 0$ is a maximum point & maximum value is $f(x = 0) = 0$.

At $x = \frac{2}{3}$, $f''\left(x = \frac{2}{3}\right) = 6\left(\frac{2}{3}\right) - 2 = 2 > 0$.

$\Rightarrow x = \frac{2}{3}$ is a minimum point & minimum value is $f\left(x = \frac{2}{3}\right) = -\frac{4}{27}$

Numerical Integration

The process of evaluating a definite integral $\int_a^b f(x)\, dx$ from a set of tabulated values (x_i, y_i); $i=0, 1, \ldots, n$ of the integrand $y = f(x)$ is called numerical integration.

Newton cote's formula (or) General Quadrature formula for equidistant coordinates

Let $I = \int_a^b y\, dx$ where $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ for $x_0, x_1, \ldots, x_n$. Let us divide the interval (a, b) into n sub intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$, $\ldots$, $x_n = x_0+nh = b$. After simplification, we get

$$I = \int_{x_0}^{x_n} y dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \tag{1}$$

which is the general quadrature formula.

By putting n = 1 in equation (1), Trapezoidal rule is obtained.

By putting n = 2 in equation (1), Simpson's $\frac{1}{3}$ rule is obtained.

By putting n = 3 in equation (1), Simpson's $\frac{3}{8}$ rule is obtained.

Trapezoidal Rule

Putting n = 1 in equation (1), we get

$$\int_{x_0}^{x_1} y \ dx = \int_{x_0}^{x_0+h} y \ dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$

$$= \frac{h}{2}\left[2y_0 + \Delta y_0 \right]$$

$$= \frac{h}{2}\left[y_0 + (y_0 + \Delta y_0) \right]$$

$$= \frac{h}{2}\left[y_0 + y_1 \right]$$

Similarly,

$$\int_{x_1}^{x_2} y \ dx = \int_{x_0+h}^{x_0+2h} y \ dx = h\left[y_1 + \frac{1}{2}\Delta y_1 \right]$$

$$= \frac{h}{2}\left[2y_1 + \Delta y_1\right]$$

$$= \frac{h}{2}\left[y_1 + \left(y_1 + \Delta y_1\right)\right]$$

$$= \frac{h}{2}\left[y_1 + y_2\right]$$

and so on.

$$\int_{x_{n-1}}^{x_n} y\ dx = \int_{x_0+(n-1)h}^{x_0+nh} y\ dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

∴ Adding all the above, we get

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + \cdots + y_{n-1}\right)\right]$$ which is called Trapezoidal Rule.

Simpson's $\frac{1}{3}$ Rule (or) Simpson's Rule

Putting n = 2 in equation (1) and neglecting the differences of higher order than second order. We get,

$$\int_{x_0}^{x_n} y \, dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right].$$

Note:

It should be noted that for applying this rule, the interval must be divided into even number of sub intervals of width h.

Simpson's $\frac{3}{8}$th Rule

Putting n = 3 in equation (1) and neglecting the higher order differences above the third, we get

$$\int_{x_0}^{x_n} y\, dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

Note:

This is not as accurate as Simpson's rule. This rule is used when the number of subintervals is a multiple of 3.

Using Trapezoidal rule, evaluate $\int_{-1}^{1}\dfrac{dx}{1+x^2}$ taking 8 intervals.

Solution

Given $y = f(x) = \dfrac{1}{1+x^2}$

Given x = -1 to 1.

$\therefore$ Length of interval = 2.

$\therefore h = \dfrac{2}{8} = 0.25$.

$\because h = \dfrac{b-a}{number\ of\ \text{int}\ ervals}$

$= \dfrac{1-(-1)}{8} = 0.25$

∴ We form a table

x	-1	-0.75	-0.5	-0.25	0	0.25	0.50	0.75	1
y	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

∴ Trapezoidal rule is $\int_{x_0}^{x_n} y \, dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$

$\int_{-1}^{1} \frac{1}{1+x^2} \, dx = \frac{0.25}{2}[(0.5 + 0.5) + 2(0.64 + 0.8 + 0.9412 + 1 + 0.9412 + 0.8 + 0.64)]$

$= \frac{0.25}{2}[1 + 2(5.7624)]$

$= 1.5656.$

Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ with $h = \dfrac{1}{6}$ by Trapezoidal rule.

Solution

Given $y = f(x) = \dfrac{1}{1+x^2}$

Given x = 0 to 1

Also $h = \dfrac{1}{6}$.

The table is

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	36/37	9/10	4/5	9/13	36/61	1/2

By Trapezoidal rule,

$$\int_{x_0}^{x_n} y\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\left(\frac{1}{6}\right)}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{31}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(3.9554)\right]$$

$$= 0.7842.$$

Evaluate $\int\limits_{4}^{5.2} \log e^x \, dx$ by using (i) Trapezoidal rule (ii) Simpson's rule (iii) Simpson's $\dfrac{3}{8}$ rule, given that h = 0.2.

Solution

Given y = f(x) = log e^x

x = 4 to 5.2, h = 0.2.

The table is

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.737	1.824	1.910	1.997	2.084	2.171	2.258

(i) By Trapezoidal rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1.737 + 2.258) + 2(1.824 + 1.910 + 1.997 + 2.084 + 2.171)\right]$$

$$= 0.3967.$$

(ii) By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

$$\int_{4}^{5.2} \log e^x\ dx = \frac{0.2}{3}\left[(1.737 + 2.258) + 4(1.824 + 1.997 + 2.171) + 2(1.910 + 2.084)\right]$$

$= 0.0666[3.995+4(5.992)+2(3.994)]$

$= 2.394.$

(iii) By Simpson's $\dfrac{3}{8}$ rule,

$$\int\limits_{x_0}^{x_n} y \ dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

$$= \frac{3(0.2)}{8}\left[(1.737 + 2.258) + 3(1.824 + 1.910 + 2.084 + 2.171) + 2(1.997)\right]$$

$$= 2.3967.$$

A curve passes through the points $(1, 2)$, $(1.5, 2.4)$, $(2, 2.7)$, $(2.5, 2.8)$, $(3, 3)$, $(3.5, 2.6)$

& $(4, 2.1)$. **Obtain Area bounded by the curve,** x **axis between** $x = 1$ **and** $x = 4$. **Also find the volume of solids of revolution by revolving this area about** $x-$ **axis.**

Solution:

$$\text{WKT, Area} = \int_a^b y\, dx = \int_1^4 y\, dx$$

$$\text{Simpson's 1/3 rule, Area} = \int_1^4 y\, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= 7.783$$

$$\text{Volume} = \pi \int_a^b y^2\, dx = \pi \int_1^4 y^2\, dx$$

To find y^2 :

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1
y^2	4	5.76	7.29	7.84	9	6.76	4.41

$$\therefore \text{ Volume } = \pi \int_1^4 y^2 dx = \pi \left\{ \frac{h}{3} \left[\left(y_0^2 + y_6^2\right) + 4\left(y_1^2 + y_3^2 + y_5^2\right) + 2\left(y_2^2 + y_4^2\right) \right] \right\} \qquad \text{[by Simpson's 1/3 rule]}$$

$$= 64.07 \text{ cubic units}$$

Compute the value of $\int\limits_{1}^{2} \dfrac{dx}{x}$ **using Simpson's 3/8 rule**

Solution: Let $y = f(x) = \dfrac{1}{x}$,

$$h = 1/3$$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3
x	1	4/3	5/3	6/3 = 2
$f(x)$	$\dfrac{1}{1} = 1$	$\dfrac{1}{4/3} = 3/4 = .75$	$\dfrac{1}{5/3} = 0.6$	$\dfrac{1}{2} = 0.5$
	y_0	y_1	y_2	y_3

WKT, Simpson's 3/8 rule is

$$\int\limits_{a}^{b} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_4) + 3\,(y_1 + y_2) + 2\,(y_3)\right]$$

i.e., $\displaystyle\int_1^2 \left(\frac{1}{x}\right) dx = \frac{3\,(1/3)}{8}\,[(1 + 0.5) + 3\,(0.75 + 0.6) + 2\,(0)]$

$$= 0.69375$$

By actual integration,

$$\int_1^2 \frac{1}{x}\,dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$